**SNSB**
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**Ergodic Theory and Additive Combinatorics**
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# Seminar 9

**(S9.1)** Let $f \in \mathcal{M}_{\mathbb{C}}(X, \mathcal{B})$ and $n \geq 1$.

(i) If $f$ is $T$-invariant (a.e.), then $S_n f = f$ (a.e.).

(ii) $S_n f \in \mathcal{M}_{\mathbb{C}}(X, \mathcal{B})$.

(iii) $S_n f = \frac{1}{n} \sum_{k=0}^{n-1} U_{T^k} f$.

(iv) For any $p \geq 1$, $f \in L^p(X, \mathcal{B}, \mu)$ (resp. $L^p_{\mathbb{R}}(X, \mathcal{B}, \mu)$) implies $S_n f \in L^p(X, \mathcal{B}, \mu)$ (resp. $L^p_{\mathbb{R}}(X, \mathcal{B}, \mu)$).

(v) For all $x \in X$, $\frac{n+1}{n} S_{n+1}(x) - S_n f(Tx) = \frac{1}{n} f(x)$.

(vi) If $f \in \mathcal{M}_{\mathbb{R}}(X, \mathcal{B})$, then $\underline{f} \circ T = \underline{f}$ and $\overline{f} \circ T = \overline{f}$.

(vii) $\int_X S_n f \, d\mu = \int_X f \, d\mu$.

(viii) If $f \in L^1_{\mathbb{R}}(X, \mathcal{B}, \mu)$ is nonnegative, then $S_n f \in L^1_{\mathbb{R}}(X, \mathcal{B}, \mu)$ is nonnegative and $\|S_n f\|_1 = \|f\|_1$.

*Proof.* (i) Obviously, since $f \circ T = f$ (a.e.) implies $f \circ T^k = f$ (a.e.) for all $k \geq 0$.

(ii) For all $k \geq 0$, we have that $f \circ T^k$ is measurable, as a composition of measurable functions. Hence, $S_n f$ is measurable as a finite sum of measurable functions.

(iii) For every $x \in X$,

$$S_n f(x) = \frac{1}{n} \sum_{k=0}^{n-1} U_{T^k} f(x) = \left( \frac{1}{n} \sum_{k=0}^{n-1} U_{T^k} f \right)(x).$$

(iv) Apply (iii) and Theorem 3.1.6.

(v)

$$\begin{aligned} S_n(Tx) &= \frac{1}{n}\sum_{k=0}^{n-1} f(T^{k+1}x) = \frac{1}{n}\sum_{k=0}^{n} f(T^k x) - \frac{1}{n}f(x) \\ &= \frac{n+1}{n}\cdot\frac{1}{n+1}\sum_{k=0}^{n} f(T^k x) - \frac{1}{n}f(x) = \frac{n+1}{n}S_{n+1}(x) - \frac{1}{n}f(x). \end{aligned}$$

Hence, $\frac{n+1}{n}S_{n+1}(x) - S_n f(Tx) = \frac{1}{n}f(x)$.

(vi) Let $x \in X$. Then

$$\begin{aligned} (\underline{f}\circ T)(x) &= \underline{f}(Tx) = \liminf_n S_n f(Tx) = \liminf_n \left(\frac{n+1}{n}S_{n+1}(x) - \frac{1}{n}f(x)\right) \\ &= \liminf_n \left(\frac{n+1}{n}S_{n+1}f(x)\right), \quad \text{since } \lim_{n\to\infty} -\frac{1}{n}f(x) = 0 \\ &= \liminf_n S_{n+1}f(x), \quad \text{since } \lim_{n\to\infty}\frac{n+1}{n} = 1 \\ &= \underline{f}(x). \end{aligned}$$

We prove similarly that $(\overline{f}\circ T)(x) = \overline{f}(x)$.

(vii) We have that

$$\int_X S_n f\, d\mu = \frac{1}{n}\sum_{k=0}^{n-1}\int_X U_{T^k} f\, d\mu = \frac{1}{n}\sum_{k=0}^{n-1}\int_X f\, d\mu,$$

by Proposition 3.1.5.

(viii) Since $U_{T^k}$ is positive for all $k$, we get that $S_n f \in L^1_{\mathbb{R}}(X, \mathcal{B}, \mu)$ is nonnegative. Apply (vii) to get that

$$\|S_n f\|_1 = \int_X S_n f\, d\mu = \int_X f\, d\mu = \|f\|_1.$$

□

**(S9.2)** Let $A, B \in \mathcal{B}$ and $n \geq 1$.

(i) $S_n\chi_A = \frac{1}{n}\sum_{k=0}^{n-1}\chi_{T^{-k}(A)}$ and $\chi_B \cdot S_n\chi_A = \frac{1}{n}\sum_{k=0}^{n-1}\chi_{T^{-k}(A)\cap B}$.

(ii) $\displaystyle\int_X S_n\chi_A = \mu(A)$.

(iii) $\displaystyle\int_X \chi_B \cdot S_n\chi_A \, d\mu = \frac{1}{n}\sum_{k=0}^{n-1} \mu(T^{-k}(A) \cap B)$.

*Proof.* Firstly, let us remark that since $\mu(X) < \infty$, $\int_X \chi_A \, d\mu = \mu(A) \leq \mu(X) < \infty$, hence $\chi_A \in L^p_{\mathbb{R}}(X, \mathcal{B}, \mu)$ for all $1 \leq p < \infty$ and $A \in \mathcal{B}$. Furthermore, $\chi_A$ is nonegative.

(i) It is an easy exercise.

(ii) Apply Proposition 4.0.7.(vii) with $f := \chi_A$.

(iii)

$$\begin{aligned}\int_X \chi_B \cdot S_n\chi_A &= \int_X \frac{1}{n}\sum_{i=0}^{n-1} \chi_{T^{-i}(A)\cap B} \, d\mu = \frac{1}{n}\sum_{i=0}^{n-1} \int_X \chi_{T^{-i}(A)\cap B} \, d\mu \\ &= \frac{1}{n}\sum_{i=0}^{n-1} \mu(T^{-i}(A) \cap B).\end{aligned}$$

□

**(S9.3)**

(i) Let $X$ be a nonempty set, $(E_n)_{n\geq 1}$ be a sequence of subsets of $X$ and $f : X \to \mathbb{R}$. Prove that

$$\lim_{n\to\infty} \chi_{\cup_{i=1}^n E_i} f = \chi_{\cup_{i\geq 1} E_i} f. \tag{D.3}$$

(ii) Let $(X, \mathcal{B}, \mu)$ be a probability space, $f \in L^1_{\mathbb{R}}(X, \mathcal{B}, \mu)$, $(E_n)_{n\geq 1}$ be an increasing sequence of measurable sets, and $E = \displaystyle\bigcup_{n\geq 1} E_n$. Prove that

$$\int_E f \, d\mu = \lim_{n\to\infty} \int_{E_n} f \, d\mu. \tag{D.4}$$

*Proof.* (i) Let

$$B_n := \bigcup_{i=1}^n E_i, \quad B := \bigcup_{i=1}^\infty E_i, \quad g_n := \chi_{B_n} f, \quad g := \chi_B f.$$

For every $x \in X$, we have two cases:

(a) $x \in B$. Then $g(x) = f(x)$ and there exists $N \geq 1$ such that $x \in E_N$. It follows that $x \in B_n$ for all $n \geq N$, hence $g_n(x) = f(x)$ for all $n \geq N$. In particular, $\displaystyle\lim_{n\to\infty} g_n(x) = f(x) = g(x)$.

(b) $x \notin B$. Then $g(x) = 0$ and $x \notin E_n$ for any $n \geq 1$. It follows that $x \notin B_n$ for any $n \geq 1$, hence $g_n(x) = 0$ for all $n \geq 1$. In particular, $\lim_{n\to\infty} g_n(x) = 0 = g(x)$.

(ii) Let $g_n := \chi_{E_n} f$ and $g := \chi_E f$. We have that

(a) $\lim_{n\to\infty} g_n = \lim_{n\to\infty} \chi_{E_n} f = \lim_{n\to\infty} \chi_{\cup_{i=1}^n E_i} f$, since $(E_n)$ is increasing. Apply now A.2.8 to conclude that $\lim_{n\to\infty} g_n = \chi_E f = g$.

(b) $|g_n| \leq |f|$ for all $n \geq 1$ and $|f| \in L^1_{\mathbb{R}}(X, \mathcal{B}, \mu)$.

We can apply Lebesgue Dominated Convergence Theorem to conclude that

$$\lim_{n\to\infty} \int_X g_n \, d\mu = \int_X g \, d\mu.$$

It follows that

$$\begin{aligned} \int_E f \, d\mu &= \int_X \chi_E f \, d\mu = \int_X g \, d\mu = \lim_{n\to\infty} \int_X g_n \, d\mu = \lim_{n\to\infty} \int_X \chi_{E_n} f \, d\mu \\ &= \lim_{n\to\infty} \int_{E_n} f \, d\mu. \end{aligned}$$

□

**(S9.4)**

**Proposition .** *Let $(X, \mathcal{B}, \mu, T)$ be a MPS. The following are equivalent*

*(i) $T$ is ergodic.*

*(ii) Whenever $f : X \to \mathbb{C}$ is measurable and $U_T f = f$, then $f$ is constant a.e..*

*(iii) Whenever $f : X \to \mathbb{C}$ is measurable and $U_T f = f$ a.e., then $f$ is constant a.e..*

*(iv) Whenever $f : X \to \mathbb{R}$ is measurable and $U_T f = f$, then $f$ is constant a.e..*

*(v) Whenever $f : X \to \mathbb{R}$ is measurable and $U_T f = f$ a.e., then $f$ is constant a.e..*

*Proof.* The following implications are trivial: (iii) ⇒ (ii), (v) ⇒ (iv), (iii) ⇒ (v), (ii) ⇒ (iv).

(v)⇒(iii) and (iv)⇒ (ii). By considering real and imaginary parts and using the fact that $U_T$ is linear, it suffices to consider $f \in \mathcal{M}_{\mathbb{R}}(X, \mathcal{B})$.

(ii)⇒(i) Let $A \in \mathcal{B}$ be such that $T^{-1}(A) = A$. Then $\chi_A$ is measurable and $U_T \chi_A = \chi_{T^{-1}(A)} = \chi_A$, so we can apply (ii) to conclude that $\chi_A$ is constant a.e. Thus, either $\chi_A = 1$ a.e., in which case $\mu(X \setminus A) = 0$ or $\chi_A = 0$ a.e., in which case $\mu(A) = 0$.

(i)⇒(v) Let $f : X \to \mathbb{R}$ be measurable with $U_T f = f$ a.e.. Hence, if $Y := \{x \in X \mid U_T f = f\}$, then $\mu(Y) = 1$. Define for each $m \geq 0$ and $k \in \mathbb{Z}$,

$$A_{m,k} = \left\{ x \in Y \mid \frac{k}{2^m} \leq f(x) < \frac{k+1}{2^m} \right\}. \tag{D.5}$$

It is easy to see that the $T$-invariance of $f$ implies that $T^{-1}(A_{m,k}) = A_{m,k}$ for all $m, k$. Furthermore, for fixed $m \geq 0$, $(A_{m,k})_{k \in \mathbb{Z}}$ is a countable family of pairwise disjoint sets satisfying $Y = \bigcup_{k \in \mathbb{Z}} A_{m,k}$. The ergodicity of $T$ implies that for every $m \geq 0$ there exists $k_m \in \mathbb{Z}$ such that $\mu(A_{m,k_m}) = 1$ and $\mu(A_{m,k}) = 0$ for all $k \neq k_m$. Let

$$A := \bigcap_{m \geq 0} A_{m,k_m}.$$

Note that

$$Y = \bigcap_{m \geq 0} \bigcup_{k \in \mathbb{Z}} A_{m,k} = \bigcup (A_{1,p_1} \cap A_{2,p_2} \cap \ldots A_{m,p_m} \cap \ldots)$$

If at least one of $p_m$'s is different from $k_m$, then the measure of the intersection is 0. Thus, we must have $\mu(A) = 1$.

Let us prove that $f$ is constant on $A$. Assume by contradiction that there are $x, y \in A$ with $f(x) - f(y) > 0$ and take $M \geq 0$ such that $2^M(f(x) - f(y)) > 1$. On the other hand $k_M \leq 2^M f(x), 2^M f(y) < k_M + 1$, hence $2^M(f(x) - f(y)) < 1$. We have got a contradiction. □